Zeros or fixed points of holomorphic functions and Cauchy's integral theorem

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A version of Bolzano's intermediate value theorem for a function f holomorphic on a bounded open neighborhood $\Omega \subset \mathbb{C}$ of the origin and continuous on $\overline{\Omega}$ was proposed in 1982 by Mau-Hsiang Shih. He showed, using Rouché's theorem, that if $\Re[\overline{z} f(z)] > 0$ on $\partial\Omega$, then f has a unique zero in Ω. Shih's condition corresponds to the one used in 1910 by Jacques Hadamard to deduce Brouwer's fixed point theorem from his version of Kronecker's index.

Another generalization of Bolzano's theorem for a function f holomorphic on the interior int P a rectangle $P \subset \mathbb{C}$ and continuous on P consists in the existence of a zero of f in P when $\Re f$ keeps opposite signs on the two vertical sides of P and $\Im f$ keeps opposite signs on the two horizontal sides of P. Those conditions correspond to the Poincaré-Miranda's conditions in dimension 2.

We show that a simple and unified methodology based only upon Cauchy's integral theorem gives a simple and unified proof of those two results, as well as the existence of a zero for holomorphic functions which behave like z^m at infinity.

Applications are given to Brouwer's fixed point theorem on a rectangle or a ball for holomorphic functions.