Esercizio n. $\boxed{4.1(a)}$ (A)

$$0 = \det(\underline{\underline{A}} - \lambda \underline{\underline{I}}) = \begin{vmatrix} (3-\lambda) & -1 & 0 \\ -1 & (3-\lambda) & 0 \\ 0 & 0 & (1-\lambda) \end{vmatrix} =$$

$$= (1-\lambda)\left[(3-\lambda)^2 - 1\right] = (1-\lambda)\left[9+\lambda^2-6\lambda-1\right] =$$

$$= (1-\lambda)(\lambda^2-6\lambda+8) \Leftrightarrow \boxed{\lambda_3 = 1}$$

$$\lambda_{1,2} = \frac{6 \pm \sqrt{36-48}}{2} = \frac{6 \pm 2}{2} = \begin{cases} 4 = \lambda_2 \\ 2 = \lambda_{\cancel{1}} \end{cases}$$

$$\underline{\underline{A}}\,\underline{v} = \lambda \underline{v}, \quad \begin{pmatrix} 3 & -1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} =$$

$$= \begin{pmatrix} 3v_1 - v_2 \\ -v_1 + 3v_2 \\ v_3 \end{pmatrix}_{i=3} = \lambda_3 \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \begin{cases} v_1 = 0 \\ v_2 = 0 \\ v_3 = 1 \end{cases}$$

$$\Rightarrow \underline{v}^3 = \hat{e}_3 = \hat{e}_3$$

Osservazione: Ricordiamo che il versore $\hat{\underline{v}}$ di $\underline{v}$ si calcola:

$$\hat{v} = |\underline{v}|^{-1}\,\underline{v}$$

3

$i=2$

$$\lambda_1\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 4v_1 \\ 4v_2 \\ 4v_3 \end{pmatrix} = \begin{pmatrix} 3v_1 - v_2 \\ -v_1 + 3v_2 \\ v_3 \end{pmatrix} \iff$$

$$\boxed{\begin{matrix}\lambda_1 = 2 \\ \lambda_2 = 4 \\ \lambda_3 = 1\end{matrix}}$$

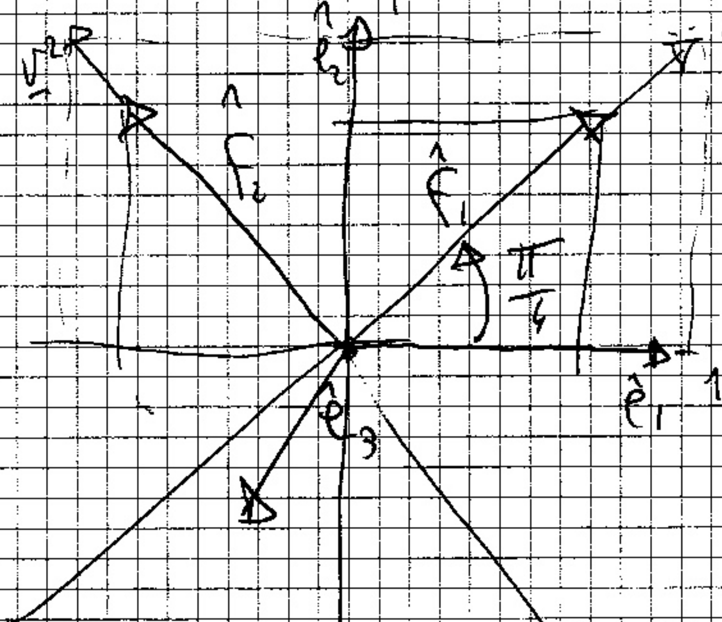
$$\begin{cases} v_3 = 0 \\ 4v_1 = 3v_1 - v_2 \\ 4v_2 = -v_1 + 3v_2 \end{cases} \qquad \boxed{v_1 = -v_2} \quad 0=0 \qquad \boxed{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \underline{v}^2}$$

$i=1$

$$\begin{cases} 2v_1 = 3v_1 - v_2 \\ 2v_2 = -v_1 + 3v_2 \\ 2v_3 = v_3 \end{cases} \iff \begin{cases} v_1 = v_2 \\ v_1 = v_2 \\ v_3 = 0 \end{cases} \qquad \boxed{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \underline{v}^1}$$

Quindi

$$\hat{f}_1 = \frac{1}{|\underline{v}^1|}\underline{v}^1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$



$$\hat{f}_2 = |\underline{v}^2|^{-1}\underline{v}^2 = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -\sqrt{2}/2 \\ \sqrt{2}/2 \\ 0 \end{pmatrix}$$

Calcoliamo ora la rotazione relativa in base all'esempio p. 2 di (12)

$$\hat{f}_1 = \cos\frac{\pi}{4}\,\hat{e}_1 + \sin\frac{\pi}{4}\,\hat{e}_2 = \frac{\sqrt{2}}{2}\hat{e}_1 + \frac{\sqrt{2}}{2}\hat{e}_2$$

$$\hat{f}_2 = -\frac{\sqrt{2}}{2}\hat{e}_1 + \frac{\sqrt{2}}{2}\hat{e}_2 = -\left(\sin\frac{\pi}{4}\right)\hat{e}_1 + \left(\cos\frac{\pi}{4}\right)\hat{e}_2$$

$$\hat{f}_3 = \hat{e}_3$$

Rotazione Associata

$$\underset{=R^{-1}}{R^T} = \begin{pmatrix} \cos\frac{\pi}{4} & \sin\frac{\pi}{4} & 0 \\ -\sin\frac{\pi}{4} & \cos\frac{\pi}{4} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Osserviamo che R^T coincide con la matrice $M^{B}_{B'}$ delle formule (1.2) ed (1.3) degli Appunti (2), che corrispondono alle formule (1.58) ed (1.61) del Ruggeri.

In particolare si ha, per l'operatore $\underline{\underline{A}}$ simmetrico che

$$\left(R^T A\right) R = \begin{pmatrix} \frac{\sqrt2}{2} & \frac{\sqrt2}{2} & 0 \\ -\frac{\sqrt2}{2} & \frac{\sqrt2}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & -1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix} R = \begin{pmatrix} \sqrt2 & \sqrt2 & 0 \\ -2\sqrt2 & 2\sqrt2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{\sqrt2}{2} & -\frac{\sqrt2}{2} & 0 \\ \frac{\sqrt2}{2} & \frac{\sqrt2}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix} = A'$$ che è diagonale con i 3 autovalori sulla diagonale stessa.

Relazione fra invarianti principali I_i e autovalori per operatori simmetrici $\underline{\underline{A}} \in Sym$.

Per il teorema spettrale si ha $\underline{\underline{A}}' = \underline{R}^T \underline{\underline{A}} \underline{R} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$

allora $I_1 = \operatorname{tr} \underline{\underline{A}} = \operatorname{tr} \underline{\underline{A}}' = \lambda_1 + \lambda_2 + \lambda_3$;

$I_3 = \det \underline{\underline{A}} = \det \underline{\underline{A}}' = \lambda_1 \lambda_2 \lambda_3$;

$$I_2 = \operatorname{tr} \underline{\underline{A}}^c = \operatorname{tr} (\underline{\underline{A}}')^c = \left| A'^{(ii)} \right| = \begin{vmatrix} \lambda_2 & 0 \\ 0 & \lambda_3 \end{vmatrix} + \begin{vmatrix} \lambda_1 & 0 \\ 0 & \lambda_3 \end{vmatrix} + \begin{vmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{vmatrix} =$$

$$= \lambda_2 \lambda_3 + \lambda_1 \lambda_3 + \lambda_1 \lambda_2 .$$

$\boxed{4.1.(d)}$: $0=\det\begin{pmatrix}(1-\lambda) & 0 & 2\\ 0 & -\lambda & 0\\ 2 & 0 & (4-\lambda)\end{pmatrix} = -\lambda\left[(1-\lambda)(4-\lambda)-4\right] =$ (1)

$= -\lambda\left(\cancel{4}-5\lambda+\lambda^2-\cancel{4}\right) = -\lambda^2(\lambda-5) \Rightarrow \lambda_1=\lambda_2=0$, $\lambda_3=5$

$\underline{0}=\left(\underline{\underline{A}}-\lambda\underline{\underline{I}}\right)\underline{v} = \begin{pmatrix}1-\lambda & 0 & 2\\ 0 & -\lambda & 0\\ 2 & 0 & (4-\lambda)\end{pmatrix}\begin{pmatrix}v_1\\ v_2\\ v_3\end{pmatrix} = \begin{pmatrix}(1-\lambda)v_1+2v_3\\ -\lambda v_2\\ 2v_1+(4-\lambda)v_3\end{pmatrix} \Rightarrow$

① $\lambda_1=0=\lambda_2$, $\begin{cases}v_1+2v_3=0\\ 0=0\\ 2v_1+4v_3=0\end{cases} \Leftrightarrow \begin{cases}v_1=-2v_3\\ v_2 \text{ qualsiasi}\end{cases} \Leftrightarrow \underline{v}^1=\begin{pmatrix}-2\\0\\1\end{pmatrix}, \underline{v}^2=\underline{e}_2$

$\hat{f}_1=\hat{v}^1=\begin{pmatrix}-2/\sqrt5\\ 0\\ 1/\sqrt5\end{pmatrix}$

② $\lambda_3=5$, $\begin{cases}-4v_1+2v_3=0\\ -5v_2=0\\ 2v_1-v_3=0\end{cases}$ $\begin{cases}v_3=2v_1\\ v_2=0\\ v_3=2v_1\end{cases} \Leftrightarrow \underline{v}^3=\begin{pmatrix}1\\0\\2\end{pmatrix}$, $\hat{f}_3=\hat{v}^3=\begin{pmatrix}1/\sqrt5\\ 0\\ 2/\sqrt5\end{pmatrix}$

Alternativamente, con la (1.66), calcolare $I_1 = 1+4=5=\mathrm{tr}(A)$

$I_3=\det(\underline{\underline{A}})=0$, $I_2=\begin{vmatrix}0&0\\0&4\end{vmatrix}+\begin{vmatrix}1&2\\2&4\end{vmatrix}+\begin{vmatrix}1&0\\0&0\end{vmatrix}=0=\mathrm{tr}(A^c)$

$\Rightarrow 0=\lambda^3-5\lambda^2+0\cdot\lambda-0=(\lambda-5)\lambda^2$ con analoghe soluzioni

$\boxed{4.1.(i)}$ $I_1=\alpha+\alpha+\alpha=3\alpha$, $I_2=\begin{vmatrix}\alpha&\alpha\\ \alpha&\alpha\end{vmatrix}\cdot 3=3(\alpha^2-\alpha^2)=0$

$I_3=3\alpha^3-3\alpha^3=0 \Rightarrow 0=\lambda^3-3\alpha\lambda^2=\lambda^2(\lambda-3\alpha)$, $\boxed{\lambda_1=\lambda_2=0, \lambda_3=3\alpha}$

$\underline{0}=\left(\underline{\underline{A}}-\lambda\underline{\underline{I}}\right)\underline{v}=\begin{pmatrix}(\alpha-\lambda)v_1+\alpha v_2+\alpha v_3\\ \alpha v_1+(\alpha-\lambda)v_2+\alpha v_3\\ \alpha v_1+\alpha v_2+(\alpha-\lambda)v_3\end{pmatrix}$, ① $\lambda_1=0=\lambda_2$, $v_3=-v_1-v_2 \Rightarrow \underline{v}^1=\begin{pmatrix}1\\0\\-1\end{pmatrix}$, $\underline{v}^2=\begin{pmatrix}0\\1\\-1\end{pmatrix}$

② $\lambda_3 = 3\alpha$, $\begin{cases} -2\alpha v_1 + \alpha v_2 + \alpha v_3 = 0 \\ \alpha(v_1 - 2v_2 + v_3) = 0 \\ \alpha(v_1 + v_2 - 2v_3) = 0 \end{cases}$, $\begin{cases} -4v_2 - 2v_3 + v_2 + v_3 = 0 \\ v_1 = 2v_2 - v_3 \\ 2v_2 + v_3 + v_2 - 2v_3 = 0 \end{cases}$ $\mapsto$ E

$-3v_2 - 3v_2$

$v_1 = 2v_2 + v_3 = 5v_2$

$v_3 = 3v_2$

$$\begin{cases} -2v_1 + v_2 + v_3 = 0 \\ v_1 = 2v_2 - v_3 \\ v_1 + v_2 = 2v_3 \end{cases}$$

$$\begin{cases} -2v_1 + v_2 + v_3 = 0 \\ v_1 = 2v_2 - v_3 \\ 2v_2 - v_3 + v_2 = 2v_3 \end{cases} \quad \begin{cases} -2v_2 + v_2 + v_2 = 0 \\ \boxed{v_1 = v_2} \\ 3v_2 = 3v_3 \end{cases} \quad \begin{cases} 0 = 0 \\ \boxed{v_1 = v_2} \\ \boxed{v_2 = v_3} \end{cases}$$

$$\underline{v}^3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \hat{f}_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \underline{v}^2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \Rightarrow \hat{f}_2 = \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$$

$$\underline{v}^1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad \hat{f}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

Osserviamo che $\underline{v}^3$ è ortogonale a $\underline{v}^1$ e $\underline{v}^2$, ma $\underline{v}^1 \cdot \underline{v}^2 = 1 \neq 0$, quindi occorre trovare un altro vettore nell'autospazio di $\lambda = 0$ ortogonale a $\underline{v}^1$, cioè si deve avere $0 = \underline{v}^1 \cdot (\beta \underline{v}^1 + \gamma \underline{v}^2) = \beta \underline{v}^1 \cdot \underline{v}^1 + \gamma \underline{v}^1 \cdot \underline{v}^2 = 2\beta + \gamma$, dunque $\gamma = -2\beta$, scegliamo $\beta = 1$ e $\gamma = -2 \cdot 1 = -2$, per cui il secondo autovettore ortogonale è $\underline{u}^2 = \underline{v}^1 - 2\underline{v}^2 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ e la base ortonormale è $B' = \{\hat{f}_1, \hat{g}_2, \hat{f}_3\}$, con $\hat{g}_2 = \hat{u}^2 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$.

Costruiamo $R^T = \left(M^{B}_{B'}\right) = \begin{pmatrix} \hat{f}_1 \\ \hat{g}_2 \\ \hat{f}_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}$,

Dunque, si ha $\left(\underline{\underline{R}}^T \underline{\underline{A}}\right) \underline{\underline{R}} =$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{3\alpha}{\sqrt{3}} & \frac{3\alpha}{\sqrt{3}} & \frac{3\alpha}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3\alpha \end{pmatrix} = A'$$

// C.V.D.

$\boxed{4.2}$ Si ha $I_1 = 3\alpha$, $I_3 = 0$, $I_2 = \begin{vmatrix} \alpha & \alpha \\ 0 & \alpha \end{vmatrix} + \begin{vmatrix} \alpha & 0 \\ 0 & \alpha \end{vmatrix} + \begin{vmatrix} \alpha & -\alpha \\ -\alpha & \alpha \end{vmatrix} = 2\alpha^2 \Rightarrow 0 = \lambda^3 - 3\alpha\lambda^2 + 2\alpha^2\lambda = \lambda\left(\lambda^2 - 3\alpha\lambda + 2\alpha^2\right)$

$\Rightarrow \lambda_1 = 0$ e $\lambda_{2,3} = \dfrac{3\alpha \pm \sqrt{9\alpha^2 - 8\alpha^2}}{2} = \dfrac{3\alpha \pm \alpha}{2} = \begin{cases} 2\alpha = \lambda_2 \\ \alpha = \lambda_3 \end{cases}$.

$$\underline{0} = \left(\underline{\underline{T}} - \lambda \underline{\underline{I}}\right) \underline{v} = \begin{pmatrix} (\alpha-\lambda) & -\alpha & 0 \\ -\alpha & (\alpha-\lambda) & \alpha \\ 0 & 0 & (\alpha-\lambda) \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} (\alpha-\lambda)v_1 - \alpha v_2 \\ -\alpha v_1 + (\alpha-\lambda)v_2 + \alpha v_3 \\ (\alpha-\lambda)v_3 \end{pmatrix}$$

1) $\lambda_1 = 0$, $\begin{cases} \alpha(v_1 - v_2) = 0 \\ \alpha(-v_1 + v_2 + v_3) = 0 \\ \alpha v_3 = 0 \end{cases}$, $\begin{cases} v_1 = v_2 \\ 0 = 0 \\ v_3 = 0 \end{cases} \Leftrightarrow \underline{v}^1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

2) $\lambda_2 = 2\alpha$, $\begin{cases} -\alpha(v_1 + v_2) = 0 \\ -\alpha(v_1 + v_2 + v_3) = 0 \\ -\alpha v_3 = 0 \end{cases}$, $\begin{cases} v_1 = -v_2 \\ 0 = 0 \\ v_3 = 0 \end{cases} \Leftrightarrow \underline{v}^2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$

3) $\lambda_3 = \alpha$, $\begin{cases} -\alpha v_2 = 0 \\ -\alpha(v_1 - v_3) = 0 \\ 0 = 0 \end{cases}$, $\begin{cases} v_2 = 0 \\ v_1 = v_3 \\ 0 = 0 \end{cases} \Leftrightarrow \underline{v}^3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

Osserviamo che $\underline{v}^1 \cdot \underline{v}^2 = 0$, ma $\underline{v}^3$ non è ortogonale a nessuno; d'altra parte il tensore T non è simmetrico, quindi, per il teorema spettrale, non è diagonalizzabile.

Infatti:

$$\hat{f}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix}, \quad \hat{f}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\1\\0\end{pmatrix}, \quad \hat{f}_3 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\end{pmatrix}$$